MATH 1A − QUIZ 7 − SOLUTIONS

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Name:____________________________

(1) *(3 points)* Use linear approximation or differentials (**but not both**) to estimate:

$$\sqrt[3]{7.98}$$

Is your answer an underestimate or an overestimate? Explain **very briefly** (roughly 1 sentence) and with the help of a graph.

Linear Approximation:

Let $f(x) = \sqrt[3]{x} = x^{\frac{1}{3}}$, $a = 8$ (because notice $\sqrt[3]{8}$ is easy to evaluate)

Then:

$$L(x) = f(8) + f'(8)(x - 8)$$

But $f(8) = \sqrt[3]{8} = 2$

And $f'(x) = \frac{1}{3}x^{-\frac{2}{3}}$, and so $f'(8) = \frac{1}{3}\frac{1}{\left(\sqrt[3]{8}\right)^2} = \frac{1}{12}$.

Therefore:

$$L(x) = 2 + \frac{1}{12}(x - 8)$$

And so:

$$\sqrt[3]{7.98} = f(7.98) \approx L(7.98) = 2+\frac{1}{12}(7.98-8) = 2+\frac{-0.02}{12} = 2-\frac{0.01}{6} = 2-\frac{1}{600} \approx 1.998333\cdots$$

Differentials:

Here $dx = 7.98 - 8 = -0.02$, and:

$$dy = \left(\frac{dy}{dx}\right) dx = \frac{1}{3}\frac{1}{\left(\sqrt[3]{8}\right)^2} \times (-0.02) = \frac{1}{12} \times (-0.02) = -\frac{1}{600}$$

Hence:

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$$\sqrt[3]{7.98} = f(7.98) = f(8) + dy = 2 - \frac{1}{600} \approx 1.998333\cdots$$

Under/Overestimate:

This is an **overestimate** because near 8, the tangent line of f at 8 is **above** the graph of f:

1A/Math 1A - Fall 2013/Quizzes/Quiz 7.png

(The picture has been exaggerated a bit for you to see the distinction between $L(7.98)$ and $f(7.98)$)

(2) *(1 point)* **TRUE/FALSE**

(a) **FALSE** $\ln(Pe^{yam}) = Pyam$, where P, a, m are constants

$$\ln(Pe^{yam}) = \ln(P) + \ln(e^{yam}) = \ln(P) + yam \neq Pyam$$

(b) **FALSE** $\frac{d}{dx}10^x = x10^{x-1}$

$$\frac{d}{dx}10^x = \ln(10)10^x$$

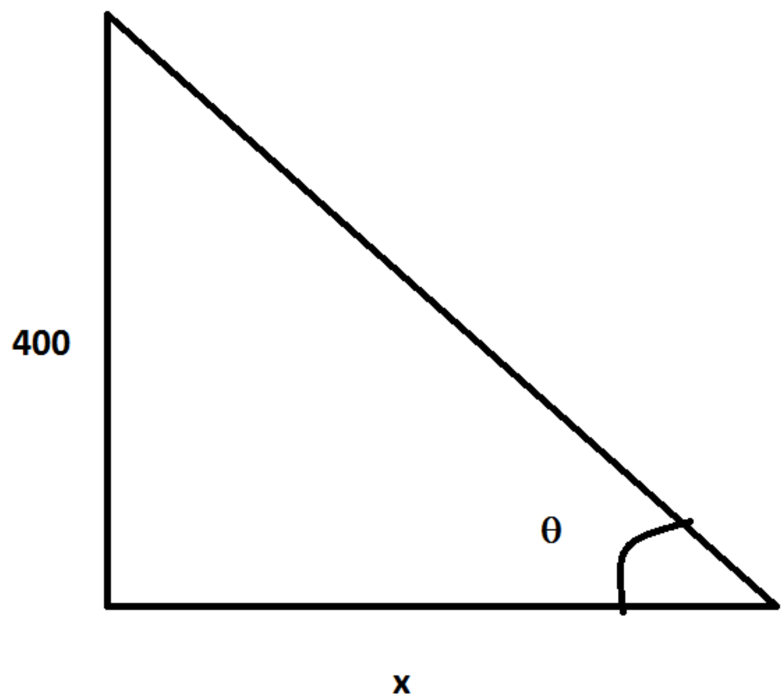
(c) **FALSE** $\frac{d}{dx}\ln(10) = \frac{1}{10}$

$\ln(10)$ is a **CONSTANT**, so you get 0.

(3) *(3 points)* The angle of elevation of the sun (that is, the angle between the sun and the floor) is decreasing at a rate of $\frac{1}{4}$ rad/h. How fast is the shadow cast by a 400 ft-tall building increasing/decreasing when that angle of elevation is $\frac{\pi}{6}$?

(1) Picture:

1A/Math 1A - Fall 2013/Quizzes/Quiz 7 Triangle.png



(2) We want to find $\frac{dx}{dt}$ when $\theta = \frac{\pi}{6}$
(3) $\tan(\theta) = \frac{400}{x}$
(4) Differentiating: $\sec^2(\theta)\frac{d\theta}{dt} = -\frac{400}{x^2}\frac{dx}{dt}$
(5) We know that $\frac{d\theta}{dt} = -0.25$ (because θ is decreasing), and $\theta = \frac{\pi}{6}$, so:

$$\sec^2(\theta) = \frac{1}{\cos^2(\theta)} = \frac{1}{\cos^2\left(\frac{\pi}{6}\right)} = \frac{1}{\left(\frac{\sqrt{3}}{2}\right)^2} = \frac{4}{3}$$

Finally, to figure out x, use the same equation as in (2), but plug in $\theta = \frac{\pi}{6}$:

$$\begin{aligned}
\tan\left(\frac{\pi}{6}\right) =& \frac{400}{x} \\
\frac{\sin\left(\frac{\pi}{6}\right)}{\cos\left(\frac{\pi}{6}\right)} =& \frac{400}{x} \\
\frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} =& \frac{400}{x} \\
\frac{1}{\sqrt{3}} =& \frac{400}{x} \\
x =& 400\sqrt{3}
\end{aligned}$$

Therefore, we get:

$$\begin{aligned}
\sec^2(\theta)\frac{d\theta}{dt} =& -\frac{400}{x^2}\frac{dx}{dt} \\
\frac{4}{3}(-0.25) =& -\frac{400}{(400)^2 \times 3}\frac{dx}{dt} \\
-\frac{1}{3} =& -\frac{1}{1200}\frac{dx}{dt} \\
\frac{dx}{dt} =& 400
\end{aligned}$$

(6) So x is **increasing** at a rate of 400 ft/h.

(4) *(3 points)* Show that:

$$-\sqrt{2} \leq \cos(x) + \sin(x) \leq \sqrt{2}$$

Let $f(x) = \cos(x) + \sin(x)$. It is enough to show that the absolute max of f is $\sqrt{2}$ and the absolute min of f is $-\sqrt{2}$. Moreover, notice that f is periodic of period 2π, so we only need to focus on the interval $[0, 2\pi]$.

(1) Endpoints: $f(0) = 1 + 0 = 1$, $f(2\pi) = 1$

(2) Critical numbers:

$$f'(x) = -\sin(x) + \cos(x) = 0 \Longrightarrow \sin(x) = \cos(x) \Longrightarrow \frac{\sin(x)}{\cos(x)} = \frac{\cos(x)}{\cos(x)} \Longrightarrow \tan(x) = 1$$

And so $x = \frac{\pi}{4}, \frac{5\pi}{4}$ (either draw a graph of $\tan$ or the trigonometric circle for that).

Now:

$$f\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) + \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \sqrt{2}$$

$$f\left(\frac{5\pi}{4}\right) = \cos\left(\frac{5\pi}{4}\right) + \sin\left(\frac{5\pi}{4}\right) = \frac{-\sqrt{2}}{2} + \frac{-\sqrt{2}}{2} = -\sqrt{2}$$

(3) Comparing:

$f(0) = 1$, $f(2\pi) = 1$, $f\left(\frac{\pi}{4}\right) = \sqrt{2}$, $f\left(\frac{5\pi}{4}\right) = -\sqrt{2}$, hence:

Absolute maximum: $f\left(\frac{\pi}{4} + 2\pi m\right) = \sqrt{2}$ (m is an integer)

Absolute minimum: $f\left(\frac{5\pi}{4} + 2\pi m\right) = -\sqrt{2}$ (m is an integer)

And we're done!